

# Monotonicity and phase diagram for multi-range percolation on oriented trees

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## Abstract

We consider Bernoulli bond percolation on oriented regular trees, where besides the usual short bonds, all bonds of a certain length are added. Independently, short bonds are open with probability  $p$  and long bonds are open with probability  $q$ . We study properties of the critical curve which delimits the set of pairs  $(p, q)$  for which there is percolation. We also show that this curve decreases with respect to the length of the long bonds.

Keywords: long range percolation, monotonicity of connectivity, critical curve

## 1 Introduction

Consider the graph having  $\mathbb{Z}^d$  as vertex set and all edges of the form  $\{x, x \pm e_i\}$  and  $\{x, x \pm k \cdot e_i\}$  for some  $k \geq 2$ . It was shown in [dLSS11] that the critical probability for Bernoulli bond percolation on this graph converges to that of  $\mathbb{Z}^{2d}$  as  $k \rightarrow \infty$ . This result, later generalized in [MT17], is a particular

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instance of Schramm’s conjecture [BNP11] that the percolation threshold for transitive graphs is a local property – which turns out to be false [BPT16]. The above convergence is conjectured to be monotone, that is, the percolation threshold for the above graph should be decreasing in the length  $k$  of long edges.<sup>1</sup>

Monotonicity questions are often intriguing for being extremely simple to ask and hard to answer. A good example [vdB] is the following: for Bernoulli bond percolation on the usual graph  $\mathbb{Z}^d$ , is the probability of the origin being connected to  $(n, 0, \dots, 0)$  monotone in  $n$ ? This question is still open, except when the parameter is close to 0 or 1 [dLPS15]. For oriented percolation on  $\mathbb{Z}_+^2$ , it has been shown [AS08] that the probability of the origin being connected to  $(m - n, m + n)$  is decreasing in  $n \in \{0, \dots, m\}$  for fixed  $m$ .

For first-passage percolation, it was conjectured [HW65] that the expected minimum travel time from  $(0, 0)$  to  $(n, 0)$  along paths contained in the cylinder  $\{(x, y) : 0 \leq x \leq n\}$  is nondecreasing in  $n$ . This question is still open, with a number of partial results [Ahl15, AW99, How01, Gou14]. In the negative direction, for first-passage percolation on  $\mathbb{Z}_+ \times \mathbb{Z}$ , it was shown [vdB83] that the expected passage time from the origin to  $(2, 0)$  is less than the expected passage time from the origin to  $(1, 0)$ . Another context where strict monotonicity is expected to happen is in the case of essential enhancements as introduced in [AG91].

In this paper we consider percolation on  $\mathbb{T}_{d,k}$ , the graph given by the oriented rooted  $d$ -regular tree bearing the usual short downward edges plus the addition of all downward edges of length  $k$ , called long edges. We let short and long edges be open independently with probability  $p$  and  $q$ , respectively. The phase diagram  $[0, 1]^2$  is decomposed in two regions: a set  $\mathcal{P}_k$  of pairs  $(p, q)$  for which a.s. there are infinite open paths, and a set  $\mathcal{N}_k$  of pairs for which a.s. there are none, see Figure 1a.

For  $p > \frac{1}{d}$  there are a.s. infinite open paths of short edges, and for  $q > \frac{1}{d^k}$  there are a.s. infinite open paths of long edges. For  $dp + d^k q \leq 1$ , a simple comparison with a branching process shows that a.s. there are no infinite

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<sup>1</sup>In support of this conjecture, simulations [AS] confirm that increasing  $k$  decreases the critical parameter, and the proof of [dLSS11, Lemma 2] shows that replacing  $k$  by a multiple does not increase it.

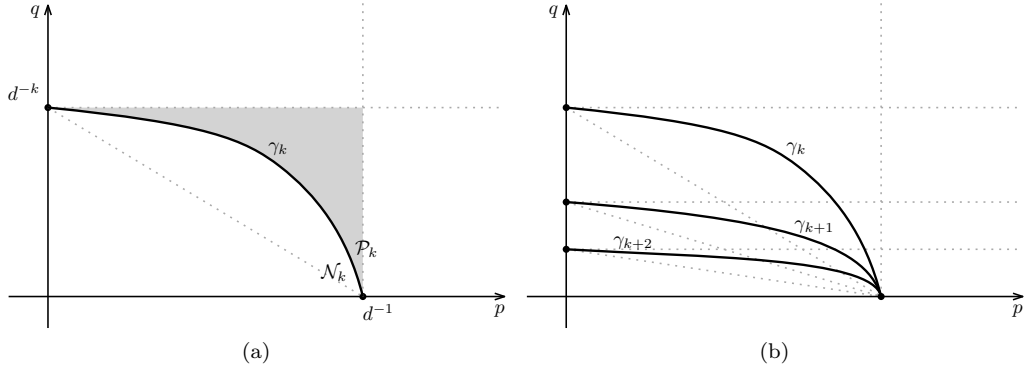


Figure 1: Phase space and critical curve separating the percolative region  $\mathcal{P}_k$  from the non-percolative region  $\mathcal{N}_k$ . (a) The curve stays between the three dotted lines. In the gray region, infinite paths necessarily use both short and long edges. (b) Critical curves for different ranges  $k$  meet at one point.

paths. By monotonicity of the process with respect to  $p$  and  $q$ , there is a *critical curve*  $\gamma_k$  joining the points  $(\frac{1}{d}, 0)$  and  $(0, \frac{1}{d^k})$  which separates  $\mathcal{N}_k$  and  $\mathcal{P}_k$ , as depicted in Figure 1a. Define

$$q_c(p, k) = \inf\{q : (p, q) \in \mathcal{P}_k\} \quad \text{and} \quad p_c(q, k) = \inf\{p : (p, q) \in \mathcal{P}_k\}. \quad (1)$$

Let  $k$  be fixed. We show that  $q_c$  is continuous and strictly decreasing in  $p$  (equivalent formulations are that  $p_c$  is strictly decreasing and continuous in  $q$ , that both  $p_c$  and  $q_c$  are continuous, or that  $\gamma$  contains neither vertical nor horizontal segments). In particular,  $\gamma_k$  is described by  $q = q_c(p, k)$  as well as by  $p = p_c(q, k)$ , and there is a non-trivial subregion of  $\mathcal{P}_k$  at which infinite paths necessarily use both long and short edges, see Figure 1a. A similar description is given in [IJvRM15] for percolation with a defect plane.

We also show that  $\gamma_{k+1}$  stays strictly below  $\gamma_k$ , and they meet only at the critical point  $(\frac{1}{d}, 0)$ . This means that  $q_c(p, k)$  is strictly decreasing in  $k$  for as long as it is positive, and analogously for  $p_c(q, k)$ , see Figure 1b.

In Section 2, we present the model and the above statements more formally.

In Section 3, we prove that  $q_c(k, p)$  is continuous and decreasing in  $p$ . In the proof, we tile  $\mathbb{T}_{d,k}$  by layers and consider a construction of the process where the state of tiles are sampled independently. We then couple configurations with different values of  $p$  and  $q$  so that some advantage in  $q$  compensates for

small decreases in  $p$  and vice-versa. Each comparison is done by finding one particular tile that makes no useful connections without extra open edges and at the same time makes all possible connections with their help. We learned this idea from [Tei06].

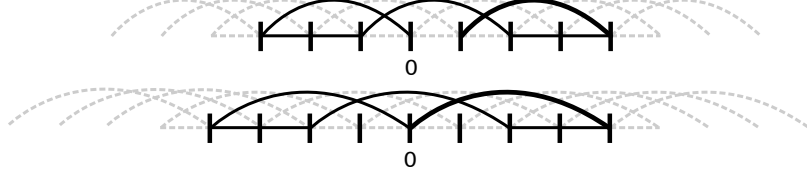
In Section 4, we show that  $q_c(p, k+1) < q_c(p, k)$ . Together with the results of Section 3, this inequality completes the previous description illustrated by Figure 1b. The proof involves a joint exploration of a percolation “cluster” in  $\mathbb{T}_{d,k}$  and a percolation cluster in  $\mathbb{T}_{d,k+1}$ . The joint exploration is an algorithm in which parts of both clusters are revealed simultaneously using coupled randomness. After each step of the algorithm is concluded, there is an injective function from the revealed portion of the cluster in  $\mathbb{T}_{d,k}$  to the one in  $\mathbb{T}_{d,k+1}$ . When trying to ensure this, one might run into collisions, that is, situations where an edge that could potentially grow the cluster in  $\mathbb{T}_{d,k}$  has as a counterpart an edge which does not grow the cluster in  $\mathbb{T}_{d,k+1}$ . The challenge is thus to design the algorithm so that collisions do not occur. We succeed in doing so by introducing a recursive procedure which alternately reveals clusters of short edges and then groups long edges, in a way that allows the comparison between the  $k$  and  $k+1$  scenarios. Note that no such comparison holds for the graph with the vertices in  $\mathbb{Z}^d$  and edges of the form  $\{x, x \pm e_i\}$  and  $\{x, x \pm k \cdot e_i\}$ , as illustrated in Figure 2. Strict inequality for  $q_c$  is obtained by extending the idea mentioned in the previous paragraph to a dynamic, hybrid construction. When revealing the state of a whole batch of long edges at once we can use the increase in  $k$  to compensate for a small decrease in  $q$ .

## 2 Definitions and results

For  $d \in \mathbb{N}$ , we denote  $[d] = \{1, \dots, d\}$ . We will make frequent use of the set

$$[d]_\star = \bigcup_{0 \leq n < \infty} [d]^n;$$

the set  $[d]^0$  is understood to consist of a single point  $o$ . Points of  $[d]_\star \setminus \{o\}$  are represented as sequences  $r = (r_1, \dots, r_n)$ . In case  $r = (r_1, \dots, r_m)$  and  $s = (s_1, \dots, s_n)$ , we define the concatenation  $r \cdot s = (r_1, \dots, r_m, s_1, \dots, s_n)$ .



*Figure 2:* Situation where the natural coupling of explorations which maps short edges to short edges and long edges to long edges leads to a “collision” in the graphs given by adding edges of length 3 and 4 to  $\mathbb{Z}$ . Dashed lines represent closed edges and full lines represent open edges. The bold edge being open increases the cluster of the first graph by one vertex, but has no effect on the cluster of the second graph.

Given  $d, k \in \mathbb{N}$ , we define the oriented graph  $\mathbb{T}_{d,k}$  as the graph with vertex set  $\mathbb{V}_{d,k} = [d]_\star$  and edge set  $\mathbb{E}_{d,k} = \mathbb{E}_{d,k}^s \cup \mathbb{E}_{d,k}^\ell$ , where

$$\begin{aligned}\mathbb{E}_{d,k}^s &= \{\langle r, r \cdot i \rangle : r \in \mathbb{V}_{d,k}, i \in [d]\}, \\ \mathbb{E}_{d,k}^\ell &= \{\langle r, r \cdot s \rangle : r \in \mathbb{V}_{d,k}, s \in [d]^k\}.\end{aligned}$$

These will be referred to as the sets of short and long edges of  $\mathbb{T}_{d,k}$ .

Consider the process in which, independently, short edges are open with probability  $p$  and long edges are open with probability  $q$ . Let  $\mathbb{P}_{p,q}$  denote the corresponding probability measure.

We define the event  $x \rightsquigarrow y$  that there exist  $x = x_0, x_1, \dots, x_{n-1}, x_n = y$  such that  $\langle x_j, x_{j+1} \rangle$  is open for all  $j < n$ , and the event  $x \rightsquigarrow \infty$  that  $x \rightsquigarrow y$  for infinitely many  $y$ . Let  $\mathcal{P}_k = \{(p, q) : \mathbb{P}_{p,q}(o \rightsquigarrow \infty) > 0\}$ ,  $\mathcal{N}_k = [0, 1]^2 \setminus \mathcal{P}_k$ , and let  $p_c(q, k)$  and  $q_c(p, k)$  be given by (1).

We prove the following monotonicity property.

**Theorem 2.1.**  $q_c(p, k+1) < q_c(p, k)$  unless  $q_c(p, k) = 0$ .

This says that  $\gamma_{k+1}$  stays under  $\gamma_k$ , and they can only intersect each other at the boundary  $\{pq = 0\}$ , except maybe where one of them contains a vertical segment. The next result rules out the latter possibility, thus completing the picture provided in Figure 1b.

**Theorem 2.2.** *For each fixed  $k \in \mathbb{N}$ , the function  $p \mapsto q_c(p, k)$  is continuous on  $[0, 1]$  and strictly decreasing on  $[0, d^{-1}]$ .*

We observe that, as a consequence of the above results, defining

$$p_c(k) = \inf\{p : (p, p) \in \mathcal{P}_k\},$$

we have  $p_c(k+1) < p_c(k)$ , as the diagonal  $\{(p, p) : 0 \leq p \leq 1\}$  intersects the critical curves  $\gamma_k$  at distinct points for different values of  $k$ . However, this conclusion can be drawn from the simpler observation that the curves  $\gamma_k$  are delimited by the dotted lines in Figure 1b.

### 3 Long and short edge compensation

The goal of this section is to prove Theorem 2.2. We will need the following elementary fact.

**Lemma 3.1.** *Let  $P_\alpha$  denote probability measures on a given finite space, parametrized by  $\alpha \in [0, 1]$ , and such that  $\alpha \mapsto P_\alpha(x)$  is continuous for every  $x$ . Let  $\kappa$  and  $y$  be such that  $P_\kappa(y) > 0$ . Then for any  $\alpha, \beta$  close enough to  $\kappa$ , there exists a coupling  $(X, Y)$  such that  $X \sim P_\alpha$ ,  $Y \sim P_\beta$  and such that, almost surely,  $X = Y$  unless  $X = y$  or  $Y = y$ .*

*Proof.* Sample the pair  $(X, Y)$  as

$$(X, Y) = \begin{cases} (z, z) & \text{w.p. } P_\alpha(z) \wedge P_\beta(z), \\ (y, z) & \text{w.p. } [P_\beta(z) - P_\alpha(z)]^+, \\ (z, y) & \text{w.p. } [P_\alpha(z) - P_\beta(z)]^+, \end{cases}$$

for  $z \neq y$ , and

$$(X, Y) = (y, y) \quad \text{w.p. } 1 - \sum_{z \neq y} P_\alpha(z) \vee P_\beta(z).$$

The last term is positive for  $\alpha$  and  $\beta$  are close to  $\kappa$  because it is positive when  $\alpha = \beta = \kappa$ . This sampling only include pairs for which  $X = Y$  unless  $X = y$  or  $Y = y$ . From the first equation we have  $\mathbb{P}(X = z) = P_\alpha(z)$  for all  $z \neq y$ , which all together imply  $\mathbb{P}(X = y) = P_\alpha(y)$ , and similarly for  $Y$ .  $\square$

We define the progeny of a vertex  $r \in \mathbb{V}_{d,k}$  as the set

$$\text{prog}(r) = \{r \cdot s \in \mathbb{V}_{d,k} : s \in [d]_\star\}.$$

The progeny of an edge is defined as the progeny of its endpoint, that is, if  $e = \langle r, s \rangle$ , then  $\text{prog}(e) = \text{prog}(s)$ .

We now turn to the proof of Theorem 2.2. Let  $\mathcal{C}_{p,q,k}$  denote the percolation “cluster” of the root in  $\mathbb{T}_{d,k}$  under the measure  $\mathbb{P}_{p,q}$ . We observe that, under this measure, the expected number of open edges with an extremity in  $o$  is equal to  $dp + d^k q$ . If such expectation is less than one, we can embed  $\mathcal{C}_{p,q,k}$  in a subcritical branching process to conclude that  $P_{p,q}(o \rightsquigarrow \infty) = 0$ . Therefore,  $q_c(p, k) \geq d^{-k} - d^{-k+1}p$ . This implies that  $q_c(p, k) > 0$  for  $p < d^{-1}$ . Since  $q_c(p, k) \leq q_c(0, k) = d^{-k}$ , we also conclude that  $p \mapsto q_c(p, k)$  is continuous at  $p = 0$ .

The proof of the theorem will thus be complete once we establish the following two facts:

$$\begin{aligned} \text{for all } p_0, q, q' \in (0, 1) \text{ with } q < q', \text{ there exist } p, p' \text{ with } p' < p_0 < p \\ \text{such that } \mathbb{P}_{p',q'}(o \rightsquigarrow \infty) \geq \mathbb{P}_{p,q}(o \rightsquigarrow \infty); \end{aligned} \quad (2)$$

$$\begin{aligned} \text{for all } q_0, p, p' \in (0, 1) \text{ with } p < p', \text{ there exist } q, q' \text{ with } q' < q_0 < q \\ \text{such that } \mathbb{P}_{p',q'}(o \rightsquigarrow \infty) \geq \mathbb{P}_{p,q}(o \rightsquigarrow \infty). \end{aligned} \quad (3)$$

Indeed, condition (2) rules out jump discontinuities in the curve  $q = q_c(p, k)$  for  $p > 0$ , and condition (3) rules out horizontal segments in this curve for  $p < d^{-1}$ .

We start the proof of (2) by introducing some notation. We let  $\bar{\mathbb{E}}_{d,k} = \bar{\mathbb{E}}_{d,k}^s \cup \bar{\mathbb{E}}_{d,k}^\ell$ , where

$$\bar{\mathbb{E}}_{d,k}^s = \{e = \langle r, s \rangle \in \mathbb{E}_{d,k}^s : r \in \cup_{n=0}^{2k-1} [d]^n\},$$

$$\bar{\mathbb{E}}_{d,k}^\ell = \{e = \langle r, s \rangle \in \mathbb{E}_{d,k}^\ell : r \in \cup_{n=0}^{2k-1} [d]^n\}.$$

Configurations in  $\bar{\Omega} = \bar{\Omega}_s \times \bar{\Omega}_\ell = \{0, 1\}^{\bar{\mathbb{E}}_{d,k}^s \cup \bar{\mathbb{E}}_{d,k}^\ell}$  are written as  $\bar{\omega} = (\bar{\omega}_s, \bar{\omega}_\ell)$ .

Given  $A \subseteq \cup_{n=0}^{k-1} [d]^n$  and  $\bar{\omega} = (\bar{\omega}_s, \bar{\omega}_\ell)$ , we define

$$J_{\bar{\omega}}(A) = \bigcup_{r \in [d]^{2k}} \left\{ \begin{array}{l} s \in \text{prog}(r) : \exists r_0, \dots, r_n \in \mathbb{V}_{d,k} \text{ so that } r_0 \in A \\ \text{and } \langle r_i, r_{i+1} \rangle \in \bar{\mathbb{E}}_{d,k}, \bar{\omega}(\langle r_i, r_{i+1} \rangle) = 1 \text{ for all } i \end{array} \right\}. \quad (4)$$

That is,  $J_{\bar{\omega}}(A)$  is the set of vertices in  $\cup_{r \in [d]^{2k}} \text{prog}(r)$  that are reachable by paths started from  $A$  and consisting only of open edges of  $\bar{\mathbb{E}}_{d,k}$ . Note that in such a path, all edges have both extremities in  $\cup_{n=0}^{2k-1} [d]^n$  except for the last one, which has only one extremity in  $\cup_{n=0}^{2k-1} [d]^n$ . In particular,  $J_{\bar{\omega}}(A) \subseteq \cup_{n=2k}^{3k-1} [d]^n$ .

Now, define the deterministic configurations  $\bar{\omega}_s^* \in \bar{\Omega}_s$  and  $\bar{\omega}_{\ell,1}^*, \bar{\omega}_{\ell,2}^* \in \bar{\Omega}_\ell$  by setting

$$\bar{\omega}_{\ell,1}^* \equiv 0, \quad \bar{\omega}_{\ell,2}^* \equiv 1 \quad \text{and} \quad \bar{\omega}_s^*(\langle r, s \rangle) = 1 \text{ if and only if } r \notin [d]^{2k-1}.$$

By Lemma 3.1, if  $p$  and  $p'$  with  $p' < p_0 < p$  are chosen sufficiently close to  $p_0$ , then there exists a coupling of configurations

$$X = (X_s, X_{\ell,1}, X_{\ell,2}), Y = (Y_s, Y_{\ell,1}, Y_{\ell,2}) \in \bar{\Omega}_s \times \bar{\Omega}_\ell \times \bar{\Omega}_\ell$$

so that the following holds:

- the values of  $X_s$ ,  $X_{\ell,1}$  and  $X_{\ell,2}$  in all edges are independent;
- $X_s$ ,  $X_{\ell,1}$  and  $X_{\ell,2}$  assign each edge to be open with respective probabilities  $p$ ,  $q$  and  $\frac{q'-q}{1-q}$ ;
- the values of  $Y_s$ ,  $Y_{\ell,1}$  and  $Y_{\ell,2}$  in all edges are independent;
- $Y_s$ ,  $Y_{\ell,1}$  and  $Y_{\ell,2}$  assign each edge to be open with respective probabilities  $p'$ ,  $q$  and  $\frac{q'-q}{1-q}$ ;
- the following event has probability one:

$$\{X = Y\} \cup \{X = (\bar{\omega}_s^*, \bar{\omega}_{\ell,1}^*, \bar{\omega}_{\ell,2}^*)\} \cup \{Y = (\bar{\omega}_s^*, \bar{\omega}_{\ell,1}^*, \bar{\omega}_{\ell,2}^*)\}. \quad (5)$$

Now take  $\bar{\omega}_s = X_s$ ,  $\bar{\omega}_\ell = X_{\ell,1}$ ,  $\bar{\omega}'_s = Y_s$ ,  $\bar{\omega}'_\ell = Y_{\ell,1} \vee Y_{\ell,2}$ .

The main observation is that each of the three events in (5) imply that, for every  $A \subseteq \cup_{n=0}^{k-1} [d]^n$ ,

$$J_{\bar{\omega}}(A) \subseteq J_{\bar{\omega}'}(A). \quad (6)$$

Indeed, on the first event we have  $\bar{\omega}' \geq \bar{\omega}$ , on the second event we have  $J_{\bar{\omega}}(A) = \emptyset$ , and on the third event  $J_{\bar{\omega}'}(A)$  contains the set of sites  $y \in \cup_{n=2k}^{3k-1} [d]^n$  that are in  $\text{prog}(x)$  for some  $x \in A$ , which always contains  $J_{\bar{\omega}}(A)$ .



Finally, with this coupling at hand, we can sample configurations  $\omega, \omega' \in \{0, 1\}^{\mathbb{E}_{d,k}}$  such that the restrictions of  $\omega$  and  $\omega'$  to sets of the form

$$\{\langle r \cdot s, y \rangle \in \mathbb{E}_{d,k} : s \in \cup_{n=0}^{2k-1} [d]^n\}$$

with  $r \in \cup_{m \in 2\mathbb{N}} [d]^{mk}$  are independent and sampled from the (appropriately translated) coupling measure. Then  $\omega$  and  $\omega'$  are distributed as  $\mathbb{P}_{p,q}$  and  $\mathbb{P}_{p',q'}$  respectively, and the “cluster” of the root in  $\omega$  is a subset of the “cluster” of the root in  $\omega'$ . This concludes the proof of (2).

We now turn to the proof of (3). As the two proofs are very similar, we now only outline the main steps of the argument.

We let  $\bar{\mathbb{E}}_{d,k}^s, \bar{\mathbb{E}}_{d,k}^\ell, \bar{\mathbb{E}}_{d,k}, \bar{\Omega}_s, \bar{\Omega}_\ell$  and  $J_{\bar{\omega}}(A)$  be the same as before. A special configuration  $\bar{\omega}^* \in \bar{\Omega}_s \times \bar{\Omega}_s \times \bar{\Omega}_\ell$  is defined as follows:

$$\bar{\omega}_{s,1}^* \equiv 0, \quad \bar{\omega}_{s,2}^* \equiv 1, \quad \bar{\omega}_\ell^*(\langle r, s \rangle) = 1 \text{ if and only if } r \in \cup_{n=k}^{2k-1} [d]^k.$$

Using Lemma 3.1, we obtain  $q' < q_0 < q$  and a coupling of  $X = (X_{s,1}, X_{s,2}, X_\ell)$  and  $Y = (Y_{s,1}, Y_{s,2}, Y_\ell)$  so that the following hold. The values of  $X_{s,1}, X_{s,2}$  and  $X_\ell$  in all edges are independent;  $X_{s,1}, X_{s,2}$  and  $X_\ell$  assign each edge to be open with respective probabilities  $p, \frac{p'-p}{1-p}$  and  $q$ ; the values of  $Y_{s,1}, Y_{s,2}$  and  $Y_\ell$  in all edges are independent;  $Y_{s,1}, Y_{s,2}$  and  $Y_\ell$  assign each edge to be open with respective probabilities  $p, \frac{p'-p}{1-p}$  and  $q'$ ; the following event has probability one:

$$\{X = Y\} \cup \{X = (\bar{\omega}_{s,1}^*, \bar{\omega}_{s,2}^*, \bar{\omega}_\ell^*)\} \cup \{Y = (\bar{\omega}_{s,1}^*, \bar{\omega}_{s,2}^*, \bar{\omega}_\ell^*)\}. \quad (7)$$

We then let  $\bar{\omega}_s = X_{s,1}, \bar{\omega}_\ell = X_\ell, \bar{\omega}'_s = Y_{s,1} \vee Y_{s,2}$  and  $\bar{\omega}'_\ell = Y_\ell$ . This coupling then guarantees (6) as before, which concludes the proof of Theorem 2.2.

## 4 Comparison of different ranges

In this section we prove Theorem 2.1.

Let  $r \in \mathbb{V}_{d,k}$  and  $s = (s_1, \dots, s_k) \in [d]^k$ , so that  $e = \langle r, r \cdot s \rangle \in \mathbb{E}_{d,k}^\ell$ . We define the trace of  $e$  to be the set of short edges

$$\text{trace}(e) = \{\langle r, r \cdot s_1 \rangle, \langle r \cdot s_1, r \cdot (s_1, s_2) \rangle, \dots, \langle r \cdot (s_1, \dots, s_{k-1}), r \cdot s \rangle\}.$$

Fix  $\omega = (\omega_s, \omega_\ell)$ , with  $\omega_s \in \{0, 1\}^{\mathbb{E}_{d,k}^s}$  and  $\omega_\ell \in \{0, 1\}^{\mathbb{E}_{d,k}^\ell}$ , and a set  $A \subseteq \mathbb{V}_{d,k}$ . We let  $\Pi(A)$  be the “cluster” of  $A$  in  $\omega$ , that is, the set of vertices of  $\mathbb{T}_{d,k}$  which can be reached by a path started from some vertex of  $A$  and consisting of directed edges which are open in  $\omega$  (note that  $\Pi(A)$  depends on  $A$  and  $\omega$  but we omit  $\omega$  from the notation; this will also be the case for further notation that we introduce). We also let  $\pi(A)$  be the cluster of  $A$  in  $\omega_s$ , that is, the set of vertices of  $\mathbb{T}_{d,k}$  that can be reached by a path started from some vertex of  $A$  and consisting of *short* edges, all of which are open in  $\omega_s$ . Note that  $A \subseteq \pi(A) \subseteq \Pi(A)$ .

We say a short edge  $e = \langle r, s \rangle \in \mathbb{E}_{d,k}^s$  is a *hub* for  $A$  (in  $\omega$ ) if the following two conditions hold:

$$\text{prog}(s) \cap \pi(A) = \emptyset \quad \text{and} \quad \text{prog}(r) \cap \pi(A) \neq \emptyset. \quad (8)$$

We let  $\sigma(A)$  denote the set of hubs for  $A$  in  $\omega$ .

**Lemma 4.1.** *Let  $\omega \in \{0, 1\}^{\mathbb{E}_{d,k}}$  and  $A \subseteq \mathbb{V}_{d,k}$ . Then,*

$$\text{the progenies } \text{prog}(e) \text{ for } e \in \sigma(A) \text{ are disjoint.} \quad (9)$$

*Further assuming that*

$$\text{there exists } r_0 \in \mathbb{V}_{d,k} : \quad A \subseteq \{r_0 \cdot s : s \in \cup_{n=0}^k [d]^n\}, \quad (10)$$

*we also have*

$$\begin{aligned} \text{for any } e = \langle r, r \cdot s \rangle \in \mathbb{E}_{d,k}^\ell \text{ such that } r \in \pi(A) \text{ and } r \cdot s \notin \pi(A), \\ \text{there exists a unique } e' \in \text{trace}(e) \cap \sigma(A) \end{aligned} \quad (11)$$

*and*

$$\begin{aligned} \Pi(A) \text{ is the disjoint union of } \pi(A) \text{ and the sets} \\ \Pi(A) \cap \text{prog}(e) \text{ for } e \in \sigma(A). \end{aligned} \quad (12)$$

*Proof.* To prove the first statement, assume that there are two edges

$$e = \langle r, s \rangle, \quad e' = \langle r', s' \rangle \in \sigma(A) : \quad \text{prog}(e) \cap \text{prog}(e') \neq \emptyset.$$

We then either have  $r, s \in \text{prog}(e') = \text{prog}(s')$  or  $r', s' \in \text{prog}(e) = \text{prog}(s)$ ; without loss of generality we assume the latter. Together with (8) applied to  $e'$ , this then implies that

$$\text{prog}(s) \cap \pi(A) \supseteq \text{prog}(r') \cap \pi(A) \neq \emptyset,$$

which contradicts (8) applied to  $e$ .

Now fix an edge  $e = \langle r, r \cdot s \rangle$  as in (11). Consider the  $k$  short edges in the trace of  $e$ . By the first statement, we know that at most one of these short edges is in  $\sigma(A)$ . In order to show that one of them is in  $\sigma(A)$ , it suffices to show that

$$\text{prog}(r) \cap \pi(A) \neq \emptyset \quad \text{and} \quad \text{prog}(r \cdot s) \cap \pi(A) = \emptyset. \quad (13)$$

The first claim of (13) follows from the fact that  $r \in \pi(A)$ ; let us prove the second. Fix  $v \in \text{prog}(r \cdot s)$ . Using (10) and the fact that  $v$  is in the progeny of the endpoint of a long edge started from  $\pi(A)$ , we see that  $v \notin A$ . Since  $r \cdot s \notin \pi(A)$  and any path of short edges from  $A$  to  $v$  contains  $r \cdot s$ , we get  $v \notin \pi(A)$ .

Statement (12) is an immediate consequence of (9) and (11).  $\square$

Again fix  $\omega \in \{0, 1\}^{\mathbb{E}_{d,k}}$  and  $A \subseteq \mathbb{V}_{d,k}$  satisfying (10). For each hub  $e \in \sigma(A)$ , we define

$$\begin{aligned} R(A, e) &= \{e' = \langle r', s' \rangle \in \mathbb{E}_{d,k}^\ell : r' \in \pi(A) \text{ and } e \in \text{trace}(e')\}, \\ \bar{S}(A, e) &= \{s' \in \mathbb{V}_{d,k} : \langle r', s' \rangle \in R(A, e) \text{ for some } r' \in \mathbb{V}_{d,k}\}, \\ S(A, e) &= \{s' \in \mathbb{V}_{d,k} : \omega(\langle r', s' \rangle) = 1 \text{ for some } \langle r', s' \rangle \in R(A, e)\}. \end{aligned}$$

Notice that  $S(A, e) \subseteq \bar{S}(A, e) \subseteq \text{prog}(e)$ . Also notice that, if  $e_1, e_2 \in \sigma(A)$  are distinct, then  $R(A, e_1)$  and  $R(A, e_2)$  are disjoint, by (9). Finally note that, for any  $e \in \sigma(A)$ , we have

$$\Pi(A) \cap \text{prog}(e) = \Pi(S(A, e)),$$

so that (12) can be restated as

$$\Pi(A) = \pi(A) \cup \left( \bigcup_{e \in \sigma(A)} \Pi(S(A, e)) \right), \quad (14)$$

where the union is disjoint.

For  $A$  satisfying (10), we now let  $\mathcal{C}_{p,q,k}(A)$  be the random set  $\Pi(A)$  when  $\omega$  is sampled from the measure  $\mathbb{P}_{p,q}$  on percolation configurations on  $\mathbb{T}_{d,k}$ . Note that  $\mathcal{C}_{p,q,k} = \mathcal{C}_{p,q,k}(\{o\})$ .

We observe that, conditioning on  $\pi(A)$ ,  $\sigma(A)$  is determined and the sets  $\Pi(S(A, e))$  are independent over  $e \in \sigma(A)$ . Indeed,  $\Pi(S(A, e))$  is determined by  $\pi(A)$  and  $\omega(e')$  for all

$$e' = \langle r', s' \rangle \text{ with } s' \in \text{prog}(e).$$

By (9), the sets of edges displayed above are disjoint for distinct choices of  $e \in \sigma(A)$ .

Guided by this consideration, we now present a recursive exploration algorithm to reveal  $\mathcal{C}_{p,q,k}(A)$ . The algorithm starts by applying the following two steps to the set  $A$ :

*Step 1.* Explore  $\pi(A)$  by revealing only the edges in  $\omega_s$  that are necessary. Note that this also reveals  $\sigma(A)$  and  $\bar{S}(A, e)$  for each  $e \in \sigma(A)$ .

*Step 2.* For each  $e \in \sigma(A)$ , reveal  $S(A, e)$ . This is the same as revealing the value of  $\omega_\ell(e')$  for each long edge  $e' \in R(A, e)$ .

Note that, if  $e = \langle r, s \rangle \in \sigma(A)$ , then  $S(A, e) \subseteq \{s \cdot s' : s' \in \cup_{n=0}^k [d]^n\}$ , that is, property (10) holds with  $A$  replaced by  $S(A, e)$ . The algorithm then proceeds by applying Steps 1 and 2 to each of the sets  $S(A, e)$ , which take the role of  $A$ . That is: in Step 1 it explores  $\pi(S(A, e))$ , which also reveals  $\sigma(S(A, e))$ , and in Step 2, for each  $e' \in \sigma(S(A, e))$ , it reveals  $S(S(A, e), e')$ . The recursion then continues to further levels. By (14), this reveals the whole cluster  $\mathcal{C}_{p,q,k}(A)$ .

We now want to look at the distributions of  $S(A, e)$  and  $\Pi(S(A, e))$  for  $e \in \sigma(A)$ . Although these distributions are easily understood, they are somewhat clumsy to describe, so we will need some more notation.

First, fix  $e = \langle r, s \rangle \in \sigma(A)$  with  $s = (s_1, \dots, s_n)$ . Define

$$\beta(A, e) = \{i \in \{1, \dots, k\} : (s_1, \dots, s_{n-i}) \in \pi(A)\},$$

it describes which ancestors of  $e$  have been reached from  $A$  using short edges and could be reach  $\text{prog}(e)$  using long edges. Note that

$$R(A, e) = \{\langle (s_1, \dots, s_{n-i}), s \cdot s' \rangle : i \in \beta(A, e), s' \in [d]^{k-i}\},$$

so that

$$\bar{S}(A, e) = \{s \cdot s' : i \in \beta(A, e), s' \in [d]^{k-i}\}.$$

Second, we define some graph isomorphisms in  $\mathbb{T}_{d,k}$ . Given  $s \in \mathbb{V}_{d,k}$ , we let  $\tau_s : \text{prog}(s) \rightarrow \mathbb{V}_{d,k}$  be the function

$$\tau_s(s \cdot s') = s', \quad s' \in [d]_\star.$$

If  $e = \langle r, s \rangle \in \mathbb{E}_{d,k}^s$ , we let  $\tau_e = \tau_s$ .

Third, we define the following probability distributions. Given  $b \subseteq \{1, \dots, k\}$ , we let

- $\mathcal{A}_{q,k}(b)$  denote the distribution of the random subset of  $\cup_{i \in b} [d]^{k-i}$  in which, independently, each point is included with probability  $q$ ;
- $\mathcal{L}_{p,q,k}(b)$  denote the law of the cluster of  $B$  in  $\mathbb{T}_{d,k}$ , where  $B$  is chosen according to  $\mathcal{A}_{q,k}(b)$ .

These objects satisfy the following identities. Let  $A \subseteq \mathbb{V}_{d,k}$  satisfy (10). Conditioning on  $\pi(A)$ , for each  $e \in \sigma(A)$  we have

$$\tau_e(S(A, e)) \stackrel{(d)}{=} \mathcal{A}_{q,k}(\beta(A, e)) \quad (15)$$

and

$$\tau_e(\Pi(S(A, e))) \stackrel{(d)}{=} \mathcal{L}_{p,q,k}(\beta(A, e)). \quad (16)$$

We finally turn to the desired comparison between  $\mathcal{C}_{p,q,k}$  for different values of the parameters. Given  $A, B \subseteq \mathbb{V}_{d,k}$ , let us write  $A \preceq B$  in case there exist  $r, s \in [d]_\star$  such that  $A \subseteq \text{prog}(r)$  and  $\tau_r(A) \subseteq \tau_s(B \cap \text{prog}(s))$ .

**Lemma 4.2.** *For any  $k \in \mathbb{N}$  and  $q \in (0, 1)$ , there exists  $q' < q$  such that the following holds. Let  $b' \subseteq \{1, \dots, k+1\}$  and  $b = b' \cap \{1, \dots, k\}$ . There exists a coupling  $(A, B)$  of random sets  $A, B \subseteq [d]_\star$  such that*

$$A \preceq B, \quad A \stackrel{(d)}{=} \mathcal{A}_{q,k}(b) \quad \text{and} \quad B \stackrel{(d)}{=} \mathcal{A}_{q',k+1}(b').$$

With this lemma at hand, fix  $p, q \in (0, 1)$  and  $k \in \mathbb{N}$ . Choose  $q'$  corresponding to  $k$  and  $q$  in Lemma 4.2. Fix  $A \subseteq \cup_{n=0}^k [d]^n$ . We compare the exploration algorithms of  $\mathcal{C}_{p,q,k}(A)$  and  $\mathcal{C}_{p,q',k+1}(A)$ . Step 1, which yields  $\pi(A)$  and  $\sigma(A)$ , has the same distribution for both clusters. Given the outcome of Step 1, it

follows from (15) and Lemma 4.2 that, for each  $e \in \sigma(A)$ , the set  $S(A, e)$  in the  $(p, q, k)$ -exploration is dominated by the corresponding set in the  $(p, q', k+1)$ -exploration. It is thus possible to couple the clusters so that  $\mathcal{C}_{p,q,k}(A)$  is embedded in  $\mathcal{C}_{p,q',k+1}(A)$ . This concludes the proof of Theorem 2.1, and it remains only to prove the previous lemma.

*Proof of Lemma 4.2.* We can assume that  $k+1 \notin b'$ , so that  $b = b'$ . In that case, for  $\hat{q} \in (0, 1)$  and  $\hat{B}$  a random subset of  $[d]_\star$ ,

$$\hat{B} \sim \mathcal{A}_{\hat{q},k+1}(b) \quad \text{if and only if} \quad \begin{array}{l} \tau_{(1)}(\hat{B}), \dots, \tau_{(d)}(\hat{B}) \text{ are independent,} \\ \text{distributed as } \mathcal{A}_{\hat{q},k}(b) \end{array} \quad (17)$$

We now define sets  $S_1^*, \dots, S_d^* \subseteq [d]_\star$  by

$$S_1^* = \emptyset, \quad S_2^*, \dots, S_d^* = \cup_{i \in b} [d]^{k-i}.$$

By Lemma 3.1, there exists  $q' < q$  and a coupling of random sets  $X_1, \dots, X_d, Y_1, \dots, Y_d \subseteq [d]_\star$  so that  $X_1, \dots, X_d$  are independent and distributed as  $\mathcal{A}_{q,k}(b)$ ,  $Y_1, \dots, Y_d$  are independent and distributed as  $\mathcal{A}_{q',k}(b)$  and the following event has probability 1:

$$\begin{aligned} \{(X_1, \dots, X_d) = (Y_1, \dots, Y_d)\} \cup \{(X_1, \dots, X_d) = (S_1^*, \dots, S_d^*)\} \\ \cup \{(Y_1, \dots, Y_d) = (S_1^*, \dots, S_d^*)\}. \end{aligned} \quad (18)$$

The desired conclusion now follows by setting

$$A = X_1, \quad B = \cup_{i \in [d]} \{i \cdot r : r \in Y_i\}. \quad \square$$

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## References

- [AG91] M. AIZENMAN, G. GRIMMETT. *Strict monotonicity for critical points in percolation and ferromagnetic models*. J Statist Phys **63**:817–835, 1991. [doi](#).
- [Ahl15] D. AHLBERG. *Asymptotics of first-passage percolation on one-dimensional graphs*. Adv in Appl Probab **47**:182–209, 2015. [doi](#).
- [AS] A. P. F. ATMAN, M. SCHNABEL. Private communication.
- [AS08] E. D. ANDJEL, M. SUED. *An inequality for oriented 2-D percolation*. In *In and out of equilibrium. 2*, vol. 60 of *Progr. Probab.*, pp. 21–30. Birkhäuser, Basel, 2008. [doi](#).
- [AW99] S. E. ALM, J. C. WIERMAN. *Inequalities for means of restricted first-passage times in percolation theory*. Combin Probab Comput **8**:307–315, 1999. [doi](#).
- [BNP11] I. BENJAMINI, A. NACHMIAS, Y. PERES. *Is the critical percolation probability local?* Probab Theory Related Fields **149**:261–269, 2011. [doi](#).
- [BPT16] D. BERINGER, G. PETE, Á. TIMÁR. *On percolation critical probabilities and unimodular random graphs*, 2016. Preprint. [arXiv:1609.07043](#).
- [dLPS15] B. N. B. DE LIMA, A. PROCACCI, R. SANCHIS. *A remark on monotonicity in Bernoulli bond percolation*. J Stat Phys **160**:1244–1248, 2015. [doi](#).
- [dLSS11] B. N. B. DE LIMA, R. SANCHIS, R. W. C. SILVA. *Critical point and percolation probability in a long range site percolation model on  $\mathbb{Z}^d$* . Stochastic Process Appl **121**:2043–2048, 2011. [doi](#).
- [Gou14] J.-B. GOUÉRÉ. *Monotonicity in first-passage percolation*. ALEA Lat Am J Probab Math Stat **11**:565–569, 2014. [pdf](#).
- [How01] C. D. HOWARD. *Differentiability and monotonicity of expected passage time in Euclidean first-passage percolation*. J Appl Probab **38**:815–827, 2001. [url](#).
- [HW65] J. M. HAMMERSLEY, D. J. A. WELSH. *First-passage percolation, sub-additive processes, stochastic networks, and generalized renewal theory*. In *Proc. Internat. Res. Semin., Statist. Lab., Univ. California, Berkeley, Calif*, pp. 61–110. Springer-Verlag, New York, 1965. [doi](#).
- [IJvRM15] G. K. ILIEV, E. J. JANSE VAN RENSBURG, N. MADRAS. *Phase diagram of inhomogeneous percolation with a defect plane*. J Stat Phys **158**:255–299, 2015. [doi](#).

- [MT17] S. MARTINEAU, V. TASSION. *Locality of percolation for abelian Cayley graphs*. Ann Probab **to appear**, 2017. [arXiv:1312.1946](#).
- [Tei06] A. Q. TEIXEIRA. *Comparison between oriented and directed bond percolation on  $Z^2$* , 2006. Oral communication.
- [vdB] J. VAN DEN BERG. Private communication.
- [vdB83] ———. *A counterexample to a conjecture of J. M. Hammersley and D. J. A. Welsh concerning first-passage percolation*. Adv in Appl Probab **15**:465–467, 1983. [doi](#).